Math 10A with Professor Stankova
Worksheet, Discussion #22; Monday, 10/16/2017
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Numerical Integration

Example

1. True **FALSE** Using the left endpoint/right endpoint/midpoint rule/trapezoid rule/Simpson's rule to approximate an integral will only give you an approximate answer and never the real answer.

Solution: These methods try to approximate the integral of a function using the area of things that we know. The left/right/midpoint methods try to approximate the function with rectangles and so if your function looks like a rectangle (constant function), then the methods will give you the exact answer. If sections of your function looks like a trapezoid (e.g. a linear function), then the trapezoid method will give you an exact answer. And finally, if your function is already a parabola, then Simpson's method will give you the exact answer since it approximates your function with a polynomial.

2. Approximate $\int_1^2 x^2 dx$ using the midpoint rule, trapezoid rule, and Simpson's rule with $n = 6$.

Solution: First we divide up our interval $[1, 2]$ into 6 intervals, each of length $\frac{1}{6}$. They are $[1, \frac{7}{6}], [\frac{7}{6}, \frac{8}{6}], [\frac{8}{6}, \frac{9}{6}], [\frac{9}{6}, \frac{10}{6}], [\frac{10}{6}, \frac{11}{6}], [\frac{11}{6}, 2]$. For midpoint, we plug in the midpoint of each interval into our function. Let $f(x) = x^2$, then we have that

$$M_6 = \frac{1}{6}\left[f\left(\frac{1+\frac{7}{6}}{2}\right) + f\left(\frac{\frac{7}{6}+\frac{8}{6}}{2}\right) + \cdots + f\left(\frac{\frac{11}{6}+2}{2}\right)\right].$$

For the trapezoid method, we plug in our function in at the left and right endpoints and take the average of those values, so we have that

$$T_6 = \frac{1}{6}\left[\frac{f(1) + f\left(\frac{7}{6}\right)}{2} + \frac{f\left(\frac{7}{6}\right) + f\left(\frac{8}{6}\right)}{2} + \cdots + \frac{f\left(\frac{11}{6}\right) + f(2)}{2}\right]$$

.

Finally, Simpson's method is a big harder. We need to divide the whole expression by 3 and then the weight which we give each height is in the pattern $1, 4, 2, 4, 2, 4, \ldots, 4, 2, 4, 1$. Doing this gives us

$$S_6 = \frac{1}{3} \cdot \frac{1}{6} \left[f(1) + 4f\left(\frac{7}{6}\right) + 2f\left(\frac{8}{6}\right) + 4f\left(\frac{9}{6}\right) + 2f\left(\frac{10}{6}\right) + 4f\left(\frac{11}{6}\right) + f(2) \right].$$

The results are shown in the table below:

n	L_n	R_n	M_n	T_n	S_n
4	1.968750	2.718750	2.328125	2.343750	2.333333
6	2.087963	2.587963	2.331019	2.337963	2.333333
10	2.185000	2.485000	2.332500	2.335000	2.333333

Problems

3. Approximate $\int_0^1 \cos(2x)dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	0.622156	0.268119	0.459419	0.445137	0.454811
6	0.568443	0.332419	0.456760	0.450431	0.454680
10	0.523940	0.382325	0.455407	0.453132	0.454653

4. Approximate $\int_0^2 e^{2x}dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	15.596437	42.395512	25.714178	28.995975	26.931923
6	18.851333	36.717383	26.309155	27.784358	26.826998
10	21.795632	32.515262	26.621245	27.155447	26.802815

5. Approximate $\int_{-1}^1 x^3dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	-0.500000	0.500000	0.000000	0.000000	0.000000
6	-0.333333	0.333333	0.000000	0.000000	0.000000
10	-0.200000	0.200000	0.000000	0.000000	0.000000

6. Approximate $\int_1^3 \ln x dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	1.007452	1.556758	1.302645	1.282105	1.295322
6	1.106594	1.472798	1.298895	1.289696	1.295721
10	1.183758	1.403480	1.296944	1.293619	1.295821

7. Approximate $\int_1^2 xe^x dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	5.968575	8.983532	7.345610	7.476053	7.389616
6	6.422771	8.432742	7.369716	7.427756	7.389167
10	6.800003	8.005986	7.382088	7.402995	7.389071

8. Approximate $\int_1^4 \sqrt{x} dx$ using Simpson's method with $n = 6$.

Solution:

n	L_n	R_n	M_n	T_n	S_n
4	4.280093	5.030093	4.672401	4.655093	4.666221
6	4.411488	4.911488	4.669245	4.661488	4.666563
10	4.514796	4.814796	4.667601	4.664796	4.666652

Error Bounds

Examples

9. True **FALSE** For calculating the error bound when using left endpoint method when approximating the integral of f on the interval $[a, b]$, we use $K_1 = f'(a)$.

Solution: We define K_1 to be the maximum of $f'(x)$ on the interval $[a, b]$. This may occur at a but that is not necessary.

10. **TRUE** False The error for an integral approximation can be negative.

Solution: Under Zvezda's definition, the error is the actual value minus the approximation. So, if your answer is an overestimate, then the error is negative (as seen in class).

11. True **FALSE** The error bound gives us what the exact error of using the different approximation techniques are.

Solution: The error bounds, as their name suggests, just allow us to bound the error. The actual error may be less than the bound (or even 0 as seen in question 1).

12. True **FALSE** The error bounds aren't helpful because they don't give us the exact error.

Solution: They are useful because they give us absolute bounds that we can use and guarantee that our approximation is good enough. In many real world applications, we need to approximate values and are fine with an error bound (e.g. length $\pm 0.05cm$) and similar to that, we can say that $\int f dx = A \pm E$, where E is the error bound.

13. How many intervals do we need to use to approximate $\displaystyle\int_1^2 x^2 dx$ within $0.001 = 10^{-3}$ using the midpoint rule? Trapezoid rule? Simpson's rule?

Solution: We take the error bound equation, set the error to be our desired bound, and solve for n. So for example, for midpoint rule, we have that $K_2 = \max|22|$ on the interval $[1, 2]$, which is just 2 so $K_2 = 4$ and we have

$$E_M = 10^{-3} = \frac{K_2(b-a)^3}{24N^2} = \frac{2}{24N^2} \implies N = \sqrt{\frac{2000}{24}} = 9.128.$$

When we are asking for the minimal number of intervals, we need an integral number and hence we take the ceiling 10 because anything greater than 9.128 gives us a good bound, and 9 does not.

The table is shown below:

Error	E_L	E_R	E_M	E_T	E_S
0.01	201	201	4	5	1
0.001	2001	2001	10	14	1
0.0001	20001	20001	30	42	1

Problems

14. How many intervals do we need to use to approximate $\int_0^1 \cos(2x)dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We have $f'(x) = -2\sin(2x)$ so $K_1 = 2$, $f''(x) = -4\cos(2x)$ so $K_2 = 4$, and $K_4 = 16$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	101	101	5	7	3
0.001	1001	1001	14	19	4
0.0001	10001	10001	42	59	6

15. How many intervals do we need to use to approximate $\int_0^2 e^{2x}dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We have $f'(x) = 2e^{2x}$ so $K_1 = 2e^4$ and $f''(x) = 4e^{2x}$ so $K_2 = 4e^4$ and $K_4 = 16e^4$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	21840	21840	86	122	12
0.001	218394	218394	271	383	21
0.0001	2183927	2183927	854	1208	36

16. How many intervals do we need to use to approximate $\int_{-1}^1 x^3dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = 3, K_2 = 6, K_4 = 0$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	601	601	15	21	1
0.001	6001	6001	46	64	1
0.0001	60001	60001	142	201	1

17. How many intervals do we need to use to approximate $\int_1^3 \ln x dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = K_2 = 1$ and $K_4 = 6$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	201	201	7	9	4
0.001	2001	2001	19	27	7
0.0001	20001	20001	59	83	11

18. How many intervals do we need to use to approximate $\int_1^2 xe^x dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = 3e^2, K_2 = 4e^2, K_4 = 6e^2$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	1109	1109	12	17	3
0.001	11085	11085	36	51	5
0.0001	110837	110837	112	158	8

19. How many intervals do we need to use to approximate $\int_1^4 \sqrt{x} dx$ within $0.001 = 10^{-3}$ using Simpson's rule?

Solution: We use $K_1 = \frac{1}{2}, K_2 = \frac{1}{4}, K_4 = \frac{15}{16}$.

Error	E_L	E_R	E_M	E_T	E_S
0.01	226	226	6	9	4
0.001	2251	2251	18	25	7
0.0001	22501	22501	54	76	12